There are two general types of waves which travel through or at the surface of the earth

- **Body waves**
- **Surface waves**

There are two types of **Body waves**
- Primary or P-waves
- Shear or S-Waves

The types of **Surface waves** that are of most interest are
- Rayleigh waves
- Love waves

Body waves can travel through the crust and mantle. P-waves can travel through the entirety of the earth, including the part of the earth’s core that is liquid. S-waves cannot near travel through the part of the earth’s core that is liquid. Surface waves are found only near the surface of the planet and are caused by the interaction of P-wave and S-waves as they reach the surface.

- P-waves are irrotational (i.e., do not cause rotation); instead, they are compressional and dilational waves similar to sound waves. The cause volumetric deformation.
  [https://my.civil.utah.edu/~bartlett/CVEEN%206330/Pwave.gif](https://my.civil.utah.edu/~bartlett/CVEEN%206330/Pwave.gif)

- S-waves cause shear or torsional deformation but do not cause any volumetric deformation in the earth’s body.
  [https://my.civil.utah.edu/~bartlett/CVEEN%206330/Swave.gif](https://my.civil.utah.edu/~bartlett/CVEEN%206330/Swave.gif)
INTRODUCTION (cont.)

- Body waves travel at velocities that depend on the stiffness and density of the crust and earth material. Because geologic materials are stiffer in compression than shear, p-waves travel faster than s-waves.

- The speed of wave travel can be used to estimate the elastic properties of the material (Young’s modulus and shear modulus).

- The interaction of inclined body waves with the stress-free surface of the earth produces surface waves. The movement caused by surface waves is restricted to a shallow zone near the surface.

- Rayleigh waves are the most important type of surface wave for earthquake engineering. In a homogeneous elastic half-space, Rayleigh waves travel slightly more slowly than s-waves and produce both vertical and horizontal particle motion that follow a retrograde elliptical pattern.

- The depth that Rayleigh waves produce significant motion is inversely proportional to the frequency of the wave. Low-frequency Rayleigh waves produce particle motions at great depths, but the motion produced by high frequency Rayleigh waves is restricted to shallow depths.

- The velocity of Rayleigh wave propagation are frequency dependent. (P and S-wave propagation velocities are frequency independent.) Low-frequency Rayleigh waves travel faster than high-frequency Rayleigh waves. Waves with frequency dependent velocities are said to be dispersive.

© Steven F. Bartlett, 2011
Love waves are another type of surface wave that develop in soft surficial layers. Love waves are also dispersive. Their velocity of propagation varies with frequency between the shear wave velocity of the surficial layer at high frequencies and the shear wave velocity of the underlying material at low frequencies.

When a body wave strikes a rigid boundary that is oriented perpendicular to its direction of travel, the wave is perfectly reflected as an identical wave traveling back in the opposite direction. The zero-displacement boundary condition requires that the stress at the boundary be twice that of the wave away from the boundary.

When a body wave strikes a stress-free boundary (like the earth's surface) oriented perpendicular to its direction of travel, the wave is reflected as an identical wave of opposite polarity traveling back in the same direction. The zero-stress boundary condition requires that the particle motion at the boundary be twice as large as the particle motion away from the boundary.

When a body wave strikes a normal boundary between two different materials, part of the wave energy is reflected and part is transmitted across the boundary. The behavior of the wave at the boundary is governed by the ratio of the specific impedances of the materials on either side of the boundary. The impedance ratio determines the amplitude and polarities of the reflected and transmitted waves.
INTRODUCTION (cont.)

- When body waves strike boundaries between different materials at angles other than 90°, part of the wave energy is reflected and part is refracted as it crosses the boundary. If the direction of particle motion is parallel to the boundary, the reflected and refracted waves will be of the same form as the incident wave. If not, new types of waves will be created. For example, an inclined p-wave that strikes a horizontal boundary will produce reflected p- and SV-waves and also refracted p- and SV-waves.

- When an inclined wave travels upward through horizontal layers that become successively softer, the portion of the wave that crosses each layer boundary will be refracted closer and closer to a vertical direction.

- The amplitude of a stress wave decreases as the wave travels through the earth's crust. There are two primary reasons that cause the attenuation (decrease) of wave amplitude. The first, material damping, is due to absorption of energy by the materials that the wave is traveling through. The second, radiation damping, results from the spreading of wave energy over a greater volume of material as it travels away from its source.

- In many cases, the ground motion at the surface can be attributed to the upward (1-0 vertical) propagation of shear waves (SH) through the soil column.

- The vertical propagation of SH waves from bedrock to the ground surface is affected by the thickness of the soil strata and the
INTRODUCTION

stress-strain and damping properties of the soil.

- Two general methods have been used to analyze the soil response to 1-D vertically propagating shear waves
  - wave equation
  - lumped masses connected by shear springs

- The widely used computer program SHAKE is formulated for 1-D wave propagation of SH waves and is based on a numeric solution to the wave equation.

WAVES IN UNBOUNDED MEDIA

- Most engineered structures can easily be idealized as assemblages of discrete masses with discrete sources of stiffness.

- Geologic materials are not as easily idealized
  - they are continuous
  - must be described by wave propagation

- Unbounded medium is an infinite medium in the direction of wave propagation

- 1-D wave propagation
  - 3 types
    - longitudinal vibration (compression/extension)
    - torsional vibration (rotation about the long axis)
    - flexural vibration (axis itself moves laterally)
1D Compressional/Tensional Wave

WAVE PROPAGATION

1-D wave propagation (cont.)

Flexural vibration has little application in soil dynamics.

Longitudinal Waves in an Infinitely Long Rod

- Infinitely long
- Linear
- Elastic
- Constrained (against straining in the radial direction)
- Derivation of 1-D wave equation (Infinitely long rod)

\[
\begin{align*}
V_x &= V_x(x, t) \\
\frac{\partial V_x}{\partial x} &= \frac{\partial V_x}{\partial x} \\
\frac{\partial^2 V_x}{\partial x^2} &= \frac{\partial^2 V_x}{\partial x^2}
\end{align*}
\]

\[
\begin{align*}
\text{u} &= \text{u}(x, t) \\
\frac{\partial u}{\partial x} &= \text{stress at right end of rod} \\
\frac{\partial u}{\partial x} &= \text{stress at left side rod} \\
\text{u} &= \text{relative displacement}
\end{align*}
\]

- Material properties \((\rho, E, \nu, A)\)
  - \(\rho\) = density
  - \(E\) = Young's modulus
  - \(\nu\) = Poisson ratio
  - \(A\) = cross-sectional area

- If rod is constrained from radial strain, then particle displacement cause by a longitudinal wave must be parallel to the axis of the rod.

- Assume cross-section \(A\) remains planar and the stress is uniformly distributed across \(A\).
1D Wave Equation for Compressional/Tensional Wave

Thursday, January 27, 2011  8:50 AM

© Steven F. Bartlett, 2011

- 1-D wave propagation (cont)
  - Derivation of 1-D wave equation (cont)
    - Write equation for dynamic equilibrium
      \[
      (V_{x_0} + \frac{\partial V_x}{\partial x}) A - V_{x_0} A = \rho A \frac{dx}{dt} \frac{\partial^2 u}{\partial t^2}
      \]
      - note \( V/A = \) stress per area = force per area
      - note unbalanced force (left side) \( \Rightarrow \) acceleration \( \cdot \) mass (right side)
    - Let \( V_x = M \dot{e}_x \)
      \[
      M = \text{constrained modulus}, \quad M = \left[ \frac{1}{(1+v) \left( 1 - 2v \right)} \right] E
      \]
      \[
      \dot{e}_x = \frac{\partial u}{\partial x}
      \]
    - Rewrite Dynamic Equilibrium
      \[
      V_{x_0} + \frac{\partial V_x}{\partial x} \frac{dx}{dt} - V_{x_0} = \rho \frac{dx}{dt} \frac{\partial^2 u}{\partial t^2}
      \]
    - Substitute \( M \dot{e}_x \) for \( V_x \)
      \[
      \frac{\partial}{\partial x} \left( M \dot{e}_x \right) = \rho \frac{dx}{dt} \frac{\partial^2 u}{\partial t^2}
      \]
    - Substitute \( \partial u / \partial x = \dot{e}_x \)
      \[
      M \frac{\partial (\partial u / \partial x)}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}
      \]
      \[
      M \frac{\partial^2 u}{\partial x^2} = \rho \frac{\partial^2 u}{\partial t^2}
      \]
1D Wave Equation for Compressional/Tensional Wave (cont.)

WAVE PROPAGATION

- 1-D wave propagation (cont.)
- Derivation of 1-D wave equation

\[ \frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho} \frac{\partial^2 u}{\partial x^2} \]

1-D wave equation

Note that the above equation can be written in an alternative form using the wave propagation speed of a p-wave, \( v_p \)

\[ v_p = \sqrt{\frac{E}{\rho}} \]

Thus

\[ \frac{\partial^2 u}{\partial t^2} = \frac{v_p^2}{\rho} \frac{\partial^2 u}{\partial x^2} \]

1-D wave equation (with p-wave velocity)

Note: \( v_p \) = wave propagation velocity of p-wave depends solely on stiffness and density. It does not depend on the amplitude of the stress wave.

Note: \( v_p \) does not equal the particle velocity of an individual point in the rod

The particle velocity is

\[ u = \frac{\partial u}{\partial t} = \varepsilon_x \frac{\partial x}{\partial t} = \frac{\varepsilon_x}{M} \frac{\partial m}{\partial t} = \frac{\varepsilon_x}{M} \frac{\partial \rho}{\partial t} \]

\[ u = \frac{\varepsilon_x}{\rho \sqrt{\rho}} \frac{\partial \rho}{\partial t} = \frac{\varepsilon_x}{\rho \sqrt{\rho}} \frac{\partial \rho}{\partial t} \]

peak particle velocity

where \( \varepsilon_m \) = wave velocity in the elastic medium
1-D wave propagation (cont.)

Example of calculating speed of primary wave propagation in an elastic medium

**Given:**

<table>
<thead>
<tr>
<th>Material</th>
<th>$G_s$</th>
<th>$M$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>7.85</td>
<td>$90.4 \times 10^6$</td>
</tr>
<tr>
<td>Vulcanized Rubber</td>
<td>1.2</td>
<td>$167 \times 10^6$</td>
</tr>
<tr>
<td>Water</td>
<td>1.0</td>
<td>$0.34 \times 10^6$</td>
</tr>
</tbody>
</table>

$G_s$ = specific gravity  
$M$ = 1-D constrained modulus

**Find:** $v_p$ for steel, rubber, water

**Solution:**

\[
v_p = \sqrt{\frac{M}{\rho}}
\]

\[
v_p = \sqrt{\frac{M \cdot g}{(G_s \rho)}}
\]

\[
v_p_{\text{steel}} = \sqrt{\left(\frac{40.4 \times 10^6 \text{ lb/ft}^2 \cdot 144 \text{ in}^2}{\text{ft}^2} \cdot \frac{32.2 \text{ ft}}{\text{s}^2}\right) \left(\frac{7.85}{(62.4 \text{ lb/ft}^3)}\right)}
\]

$v_p_{\text{steel}} = 1955.6 \text{ ft/s}

Repeating above equation for rubber & steel gives:

$v_p_{\text{rubber}} = 10163.1 \text{ ft/s}

v_p_{\text{water}} = 5026 \text{ ft/s}
1D Torsional (Shear) Wave

- 1D wave propagation (cont.)
- Torsional Waves in an Infinitely Long Rod
  - material properties \((\rho, G, \nu, A)\)

\[
\begin{align*}
\left( T_{x_0} + \frac{\partial T}{\partial x} \right) - T_{x_0} &= \rho J dx \frac{\partial^2 \theta}{\partial t^2} \\
\text{unbalanced external torque} &= \text{inertial torque} \\
J &= \text{polar moment of inertia of the rod}.
\end{align*}
\]

- The above simplifies to:

\[
\frac{\partial T}{\partial x} = \rho J \frac{\partial^2 \theta}{\partial t^2}
\]

- Torque can be related to rotation by:

© Steven F. Bartlett, 2011
1D Wave Equation for a Torsional Wave

T = G J \frac{\partial \theta}{\partial x}

G = shear modulus

Thus, the torsional wave equation can be written as:

\[ \frac{\partial^2 \theta}{\partial t^2} = \frac{G \partial^2 \theta}{\partial x^2} = \nu_s^2 \frac{\partial^2 \theta}{\partial x^2} \]

where \( \nu_s \) = shear wave propagation velocity

\[ \nu_s = \sqrt{G/\rho} \]

and

\[ G = \rho \nu_s^2 \]

The above equation is extremely useful. It says that if we can measure the speed of shear wave propagation and if we know the density of the rock or soil, it is possible to calculate the shear modulus of the rock or soil layer.

Example of calculating speed of shear wave propagation

**Given:**

<table>
<thead>
<tr>
<th>Material</th>
<th>( G ) (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel</td>
<td>11.5 x 10^6</td>
</tr>
<tr>
<td>Vulcanized Rubber</td>
<td>0.167 x 10^6</td>
</tr>
<tr>
<td>Water</td>
<td>0</td>
</tr>
</tbody>
</table>
Speed of S wave Propagation

Thursday, January 27, 2011

Page 12 of

**WAVE PROPAGATION**

1-D Wave Propagation (cont.)

Find: $V_s$ for steel, rubber, water

Solution:

$$V_s = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{G_s g}{\gamma_w}}$$

where $g$ = acceleration gravity
$G_s$ = specific gravity
$\gamma_w$ = unit weight of water

for steel

$$V_s = \left(\frac{145 \times 10^4 \text{ lb/} \text{ in}^2 \cdot \frac{14.4 \text{ in}^2}{\text{ft}^2} \cdot 32.2 \text{ ft/s}^2}{7.85 \cdot 62.4 \text{ lb/ft}^2}\right)^{1/2}$$

$$V_s = 10434 \text{ ft/s}$$

for rubber

$$V_s = 3216 \text{ ft/s}$$

for water

$$V_s = 0$$

Note that shear waves cannot propagate through an inviscid fluid.

Solution to 1-D wave equation

Form of wave equation

$$\frac{\partial^2 u}{\partial t^2} = V^2 \frac{\partial^2 u}{\partial x^2}$$

© Steven F. Bartlett, 2011
Solution for 1-D Wave Equation

Thursday, January 27, 2011
8:50 AM

Solution to 1-D wave equation (cont.)

\[ v = \text{wave propagation velocity} \]

- Solution can be written in form

\[ u(x,t) = f(vt-x) + g(vt+x) \]

- Where \( f \) and \( g \) are arbitrary functions of \( vt-x \) and \( vt+x \) that satisfy the wave eq. on the previous page.

- The solution describes a displacement wave \( f(vt-x) \) traveling at velocity \( v \) in the positive x-direction and another wave \( g(vt+x) \) traveling at the same speed in the negative x-direction.

- Solution also describes a standing wave (shape of wave does not change with position or with time).

- Rod subjected to steady-state harmonic stress (forced vibration)

\[ \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left( \omega_0 \cos \omega_0 t \right) \]

Where \( \omega_0 = \text{amplitude of stress wave} \)

\[ \omega = \text{circular frequency of applied loading} \]

Then

\[ u(x,t) = A \cos (\omega t - kx) + B \cos (\omega t + kx) \]

Where the first term and second terms describe harmonic waves propagating in the negative and positive x-directions, respectively.
* Solution to 1-D wave equation (cont.)

where \( k \) is the wave number

\[
k = \frac{\omega}{v}
\]

\[
k = \frac{2\pi}{\lambda}
\]

where \( \lambda \) = wavelength

\[
\lambda = \frac{v}{\bar{T}} = \frac{v}{\frac{2\pi}{\omega}} = \frac{2\pi v}{\omega} = 2\pi \frac{v}{k}
\]

where \( \bar{T} \) = period of applied loading

\( \bar{T} = \frac{1}{f} \)

* Equation 13-1 indicates that displacement varies harmonically with respect to both time and position

\( u \) vs. time

\[
T = \frac{2\pi}{\omega}
\]

\( u \) vs. \( x \)

\[
\lambda = \frac{2\pi}{k}
\]

© Steven F. Bartlett, 2011
WAVE PROPAGATION

Solution to 1-D wave equation (cont.)

Complex notation

\[ u(x,t) = Ce^{i(\omega t - kx)} + De^{i(\omega t + kx)} \]

Example Problem

**GIVEN:**
- \( V_p \) steel = 13556 ft/s
- \( V_p \) vulcanized rubber = 10163 ft/s
- \( V_s \) steel = 10434 ft/s
- \( V_s \) vulcanized rubber = 3216 ft/s

**FIND:** wavelength \( \lambda \) for p-wave & s-wave.

**ASSUME:** harmonic wave at a frequency of 10 Hz

**SOLUTION:**

\[ \lambda = \frac{V_p}{f} = \frac{13556 \text{ ft/s}}{10/\text{s}} = 1356 \text{ ft} \quad \text{steel} \]

\[ \lambda = \frac{V_s}{f} = \frac{10163 \text{ ft/s}}{10/\text{s}} = 1016 \text{ ft} \quad \text{rubber} \]

\[ \lambda = \frac{V_p}{f} = \frac{10434 \text{ ft/s}}{10/\text{s}} = 1043 \text{ ft} \quad \text{steel} \]

\[ \lambda = \frac{V_s}{f} = \frac{3216 \text{ ft/s}}{10/\text{s}} = 322 \text{ ft} \quad \text{rubber} \]
3-D Wave Propagation

Review of Stress Notation

- Note that stresses on opposing faces of cube have opposite directions of action otherwise rotation would occur.

- Note also that \( \tau_{xy} = \tau_{yx} \), \( \tau_{xz} = \tau_{zx} \) and \( \tau_{yz} = \tau_{zy} \) otherwise rotation would occur.

- Note that only six independent components of stress are required to describe this 3-D system.

\[
\begin{align*}
\tau_x &= \tau_{xx} \\
\tau_y &= \tau_{yy} \\
\tau_z &= \tau_{zz} \\
\tau_{xy} &= \tau_{yx} \\
\tau_{xz} &= \tau_{zx} \\
\tau_{yz} &= \tau_{zy}
\end{align*}
\]

Note: \( \tau_{xx} \) plane upon which the stress acts (normal stress act normal to this plane and shear stress act parallel to this plane).
Review of Strain Notation

2-D Plane Strain (with translation, distortion, rotation)

Relationships and Definitions

- $u = \text{displacement in x direction (translation)}$
- $v = \text{displacement in y direction (translation)}$
- $\alpha_1$, $\alpha_2$ are the partial shear strains (in radians) due to change in angle (distortion) of $QP$ and $SP$
- $\chi = \alpha_1 + \alpha_2$ ("engineering shear strain")
- $\gamma_{xy} = \alpha_1$, $\alpha_2$ ("Cauchy shear strain")
- For small rotations, $\alpha_1$ and $\alpha_2 \ll 1$, thus $\tan \alpha_1 \approx \alpha_1$ and $\tan \alpha_2 \approx \alpha_2$ and therefore
  - $\Delta \approx \frac{dy}{dx}$
  - $\Delta_2 \approx \frac{dy}{dy}$
  - $\gamma_{xy} \approx \frac{dy}{dx} + \frac{du}{dy}$
- Rotation $\Omega = (\alpha_1 - \alpha_2)/2$

© Steven F. Bartlett, 2011
3-D WAVE PROPAGATION

- Review of Strain Notation (cont.)

- 3-D strain

\[
\begin{align*}
\varepsilon_{xx} &= \frac{du}{dx} \\
\varepsilon_{yy} &= \frac{dv}{dy} \\
\varepsilon_{zz} &= \frac{dw}{dz} \\
\gamma_{xy} &= \frac{dv}{dx} + \frac{du}{dy} \\
\gamma_{yz} &= \frac{dw}{dy} + \frac{du}{dz} \\
\gamma_{zx} &= \frac{dw}{dx} + \frac{dv}{dz}
\end{align*}
\]

- Note: \( \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz} \) represent extension or compression parallel to \( x, y, z \) axes and are called normal strains.

- Note: \( \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \) represent shear strains in the \( xy, yz, zx \) planes, respectively.

- Note: \( \gamma_{xy} = 2 \varepsilon_{xy} \) where \( \varepsilon_{xy} \) is sometimes used in some textbooks as an alternative notation for shear strain in tensor notation.

- 3-D Rigid body rotations

\[
\begin{align*}
\Omega_x &= \frac{1}{2} \left( \frac{dw}{dz} - \frac{du}{dx} \right) \quad \text{(rotation about } x\text{-axis)} \\
\Omega_y &= \frac{1}{2} \left( \frac{dw}{dx} - \frac{dv}{dz} \right) \quad \text{(rotation about } y\text{-axis)} \\
\Omega_z &= \frac{1}{2} \left( \frac{dv}{dx} - \frac{du}{dy} \right) \quad \text{(rotation about } z\text{-axis)}
\end{align*}
\]

- Stress-strain Relations (Hooke’s law)

  - Stress and strain are proportional in a linear elastic body.
Hooke's Law (cont.)

3-D Wave Propagation

- Hooke's Law (cont.)

General form:

\[ \nabla_{xx} = C_{11} \varepsilon_{xx} + C_{12} \varepsilon_{yy} + C_{13} \varepsilon_{zz} + C_{44} \varepsilon_{xy} + C_{55} \varepsilon_{yz} + C_{66} \varepsilon_{zx} \]

\[ \nabla_{yy} = C_{21} \varepsilon_{xx} + C_{22} \varepsilon_{yy} + C_{23} \varepsilon_{zz} + C_{44} \varepsilon_{yx} + C_{55} \varepsilon_{zy} + C_{66} \varepsilon_{zx} \]

\[ \nabla_{zz} = C_{31} \varepsilon_{xx} + C_{32} \varepsilon_{yy} + C_{33} \varepsilon_{zz} + C_{44} \varepsilon_{zx} + C_{55} \varepsilon_{zy} + C_{66} \varepsilon_{yx} \]

\[ T_{xy} = C_{44} \varepsilon_{xx} + C_{45} \varepsilon_{yy} + C_{46} \varepsilon_{zz} + C_{44} \varepsilon_{yx} + C_{45} \varepsilon_{zy} + C_{46} \varepsilon_{zx} \]

\[ T_{yz} = C_{55} \varepsilon_{xx} + C_{56} \varepsilon_{yy} + C_{53} \varepsilon_{zz} + C_{54} \varepsilon_{yx} + C_{55} \varepsilon_{zy} + C_{56} \varepsilon_{zx} \]

\[ T_{zx} = C_{66} \varepsilon_{xx} + C_{63} \varepsilon_{yy} + C_{65} \varepsilon_{zz} + C_{64} \varepsilon_{yx} + C_{65} \varepsilon_{zy} + C_{66} \varepsilon_{zx} \]

- Note the \( c \) coefficients are elastic constants for the material.

- Because elastic strain energy must be a unique function of strain,

\[ C_{ij} = C_{ji} \]

which reduces the number of coefficients from 36 to 21.

- If the material is isotropic, then

\[ C_{12} = C_{21} = C_{13} = C_{31} = C_{23} = C_{32} = \lambda \]

\[ C_{44} = C_{55} = C_{66} = \mu \]

\[ C_{11} = C_{22} = C_{33} = \lambda + 2\mu \]

- Hooke's law for an isotropic, linear, elastic material can be expressed with two constants, \( \lambda \) and \( \mu \), which are Lamé constants.
3-D WAVE PROPAGATION

- Hooke’s Law (cont.)

\[ \nabla_{xx} = \lambda \varepsilon + 2\mu \varepsilon_{xx} \]
\[ \nabla_{yy} = \lambda \varepsilon + 2\mu \varepsilon_{yy} \]
\[ \nabla_{zz} = \lambda \varepsilon + 2\mu \varepsilon_{zz} \]
\[ \nabla_{xy} = \mu \gamma_{xy} \]
\[ \nabla_{yz} = \mu \gamma_{yz} \]
\[ \nabla_{zx} = \mu \gamma_{zx} \]

where \( \varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \) = volumetric strain

- Elastic material properties from Lamé’s constants

Young’s modulus: \( E = \frac{\mu (3\lambda + 2\mu)}{\lambda + \mu} \)

Bulk modulus: \( K = \frac{\lambda + 2\mu}{3} \)

Shear modulus: \( G = \mu \)

Poisson’s ratio: \( \nu = \frac{\lambda}{2(\lambda + \mu)} \)

- Equations of Motion for 3-D elastic solid

- Derived in same way as 1-D rod
- Dynamic equilibrium must be considered in 3 directions
- Consider stress in an infinitesimal cube aligned with sides parallel to \( x, y, z \) axes
Derivation of Wave Equation for 3D Elastic Body

3-D Wave Propagation

Equations of Motion for 3-D Elastic Solid (Cont.)

Assume average stress on each face is represented by a stress vector at the center of face.

\[ \tau_{xx} + \frac{\partial \tau_{xx}}{\partial z} \, dz \]
\[ \tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} \, dy \]
\[ \tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} \, dz \]
\[ \nabla \times \frac{\partial \tau_{xx}}{\partial x} \, dx \]

Writing the equation of dynamic equilibrium for x-direction only

\[ \rho \, dx \, dy \, dz \, \frac{\partial^2 u}{\partial t^2} = \left( \nabla \tau_{xx} \frac{\partial \tau_{xx}}{\partial x} \right) \, dy \, dz \, - \nabla \tau_{xx} \, dy \, dz \]
\[ + \left( \tau_{xy} + \frac{\partial \tau_{xy}}{\partial y} \right) \, dx \, dz \, - \tau_{xy} \, dx \, dz \]
\[ + \left( \tau_{xz} + \frac{\partial \tau_{xz}}{\partial z} \right) \, dx \, dy \, - \tau_{xz} \, dx \, dy \]

This simplifies to

\[ \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\tau_{xy}}{\partial y} + \frac{\tau_{xz}}{\partial z} \quad (\text{x-direction}) \]
3-D WAVE PROPAGATION

**Equations of Motion 3-D (cont.)**

- Repeating for y and z directions

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}
\]

\[
\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}
\]

**Equations of motion in terms of strain-displacement**

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{2}{\rho} \left( \lambda + 2\mu \right) \frac{\partial \varepsilon}{\partial x} + \frac{2}{\rho} \mu \frac{\partial \gamma_{xy}}{\partial y} + \frac{2}{\rho} \mu \frac{\partial \gamma_{xz}}{\partial z}
\]

\[
\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \gamma_{xy} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}, \quad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\]

- This produces

\[
\rho \frac{\partial^2 u}{\partial t^2} = \left( \lambda + \mu \right) \frac{\partial \varepsilon}{\partial x} + \mu \nabla^2 u \quad (x\text{-direction only})
\]

Where \( \nabla^2 \) is a Laplacian operator

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]

- For y and z directions

\[
\rho \frac{\partial^2 v}{\partial t^2} = \left( \lambda + \mu \right) \frac{\partial \gamma_{xy}}{\partial y} + \mu \nabla^2 v \quad (y\text{-direction})
\]

\[
\rho \frac{\partial^2 w}{\partial t^2} = \left( \lambda + \mu \right) \frac{\partial \gamma_{xz}}{\partial z} + \mu \nabla^2 w \quad (z\text{-direction})
\]
3-D Wave Propagation

* **Solution for 3-D Equations of Motion**
  - The previous equations of motion can be manipulated to produce two wave equations
  - body waves travel through unbounded solid

* **First type of wave**
  - Differentiate Eq. 7-1, 7-2, 7-3 and add results together
  \[
  \rho \left( \frac{\partial^2 \varepsilon_{xx}}{\partial t^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial t^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial t^2} \right) = (\lambda + \mu) \left( \frac{\partial^2 \varepsilon_{xx}}{\partial x^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial y^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial z^2} \right) + \mu \left( \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} + \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} + \frac{\partial^2 \varepsilon_{xz}}{\partial z \partial x} \right)
  \]
  - Can be written as
  \[
  \rho \frac{\partial^2 \varepsilon}{\partial t^2} = (\lambda + \mu) \nabla^2 \varepsilon + \mu \nabla^2 \varepsilon
  \]

* **Rearrange into wave equation**
  \[
  \frac{\partial^2 \varepsilon}{\partial t^2} = \frac{\lambda + 2\mu}{\rho} \nabla^2 \varepsilon
  \]
  - \( \varepsilon \) is volumetric strain (deformation that involves no shearing or rotation)
  - The above equation describes an irrotational or dilatational wave

  where \( \sqrt{\frac{\lambda + 2\mu}{\rho}} \) = velocity of propagation

  \( V_p \) (velocity primary wave)

  where \( V_p = \sqrt{\frac{G(2-2\nu)}{\rho(1-2\nu)}} \)
3-D WAVE PROPAGATION

* Solution for 3-D Equations of Motion
  
  * Second type of wave (Distortion or shear wave)
    
    \[ \rho \frac{\partial^2}{\partial t^2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \mu \nabla^2 \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \]
  
  * Note to obtain the solution above, \( \epsilon \) is eliminated by differentiating equation 7-2 with respect to \( z \) and equation 7-3 with respect to \( y \) and subtracting one from the other:
    
    \[ \rho \frac{\partial^2}{\partial t^2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) = \mu \nabla^2 \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \]
  
  Recalling that (p. 2, l p. 3)

\[ \Omega_x = \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \]

for rigid-body rotation about the \( x \)-axis

then the above equation can be written as:

\[ \frac{\partial^2 \Omega_x}{\partial t^2} = \frac{\mu}{\rho} \nabla^2 \Omega_x \]

Wave Equation

Distortional Wave

The above equation describes an equal volume or distortional wave about the \( x \)-axis.

Similar equations can be written for the \( y \) and \( z \) axes.
3-D Wave Propagation

- Solution for 3-D Equations of Motion
  - Distortion of shear waves (cont.)
    - Wave propagation speed ($V_s$)
      \[ V_s = \sqrt{\frac{\mu}{\rho}} = \sqrt{\frac{G}{\rho}} \]
      where $\mu = G = \text{shear modulus}$
    - For S-wave, the direction of particle motion is perpendicular to the direction of wave propagation.
    - $SH$-waves - particle motion is in a horizontal plane.
      \[ \text{particle motion} \]
      \[ \text{direction of wave propagation} \]
    - $SV$-waves - particle motion is in a vertical plane.
      \[ \text{particle motion} \]
      \[ \text{direction of wave propagation} \]
    - A S-wave with arbitrary particle motion can be represented by the vector sum of its $SH$ and $SV$ components.
    - Relation between $V_p$ and $V_s$
      \[ \frac{V_p}{V_s} = \sqrt{\frac{2-2v}{1-2v}} \quad V = \text{Poisson's ratio} \]
Waves in a Semi-Infinite Body

- Elastic half-space

- The earth's surface is not an elastic half-space, but for engineering purposes, can be idealized as such.

- The boundary condition of the free surface causes other types of waves to form at the free surface.

- These are known as surface waves and are located very near the earth's surface.

- Surface waves attenuate more slowly than body waves. (Attenuate means that the amplitude diminishes with distance from the source).

Types of surface waves:

- Rayleigh waves
- Love waves

Rayleigh Waves:

- Plane wave that travels in the x-direction with zero particle displacement in the y-direction.
3-D WAVE PROPAGATION

* Rayleigh Waves (cont.)

- describe displacements in the $x$ and $z$-directions with functions

$$\varepsilon_0 = \frac{\partial E}{\partial x} + \frac{\partial E}{\partial z}$$

$$w = \frac{\partial E}{\partial z} - \frac{\partial E}{\partial x}$$

- volumetric strain or dilation $\bar{\varepsilon}$ is given by

$$\bar{\varepsilon} = \varepsilon_{xx} + \varepsilon_{zz}$$

$$\bar{\varepsilon} = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \frac{1}{2} \left( \frac{\partial E}{\partial x} + \frac{\partial E}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{\partial E}{\partial z} \right) \left( \frac{\partial E}{\partial x} \right)$$

$$\bar{\varepsilon} = \frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial z^2}$$

$$\bar{\varepsilon} = \nabla^2 \phi$$

© Steven F. Bartlett, 2011
3-D WAVE PROPAGATION

Rayleigh Waves (cont.)

\[ 2 \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial}{\partial z} \left( \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial z} \right) - \frac{\partial}{\partial x} \left( \frac{\partial \Phi}{\partial z} - \frac{\partial \Psi}{\partial x} \right) \]

\[ = \frac{\partial^2 \Psi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial x^2} \]

\[ 2 \Omega_y = \nabla^2 \Psi \quad \text{rotation} \]

Use of the potential functions \( \Phi, \Psi \) allows the separation of the effects of dilation and rotation respectively.

Rayleigh waves can be thought of \( p- \) and \( s- \) waves (\( SV \) waves for this case) that satisfy certain boundary conditions.

Substitution of the expressions \( u \) and \( w \) into the equations of motion:

\[ \rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu) \frac{\partial^2 \Phi}{\partial x^2} + \mu \nabla^2 u \quad \text{(eq. motion)} \]

\[ \rho \frac{\partial}{\partial x} \left( \frac{\partial^2 \Phi}{\partial t^2} \right) + \rho \frac{\partial}{\partial z} \left( \frac{\partial^2 \Psi}{\partial t^2} \right) = (\lambda + 2\mu) \frac{\partial}{\partial x} \left( \nabla^2 \Phi \right) 
+ \mu \frac{\partial}{\partial z} \left( \nabla^2 \Psi \right) \]

Also:

\[ \rho \frac{\partial}{\partial z} \left( \frac{\partial^2 \Phi}{\partial t^2} \right) - \rho \frac{\partial}{\partial x} \left( \frac{\partial^2 \Psi}{\partial t^2} \right) = (\lambda + 2\mu) \frac{\partial}{\partial z} \left( \nabla^2 \Psi \right) 
- \mu \frac{\partial}{\partial x} \left( \nabla^2 \Psi \right) \]
3-D WAVE PROPAGATION

- Rayleigh Waves (cont.)
  
  Solving the two previous equations simultaneously for $\frac{\partial^2 \Phi}{\partial t^2}$ and for $\frac{\partial^2 \psi}{\partial t^2}$ yields

  \[ \frac{\partial^2 \Phi}{\partial t^2} = \lambda + 2\mu \nabla^2 \Phi = v_p^2 \nabla^2 \Phi \]  
  \[ \text{Rayleigh wave eq.} \]

  \[ \frac{\partial^2 \psi}{\partial t^2} = \mu \nabla^2 \psi = v_s^2 \nabla^2 \psi \]  
  \[ \text{Rayleigh wave eq.} \]

- Rayleigh Wave velocities

  Rayleigh waves are of interest in geotechnical engineering because their velocity is used to estimate the stiffness of surficial soils. The method to do this is spectral analysis of surface waves (SASW).

  The Rayleigh wave velocity is often expressed as the ratio of this velocity to the $S$-wave velocity

  \[ K_{RS} = \frac{V_R}{V_S} = \frac{\omega}{V_s \lambda_R} \]

  where

  $V_R$ = Rayleigh wave velocity

  $V_S$ = Shear wave velocity

  $\lambda_R$ = wave number = $\frac{2\pi}{\lambda}$ = wavelength

© Steven F. Bartlett, 2011
3-D WAVE PROPAGATION

- Rayleigh Waves (cont.)
- Rayleigh wave velocities
  - The Rayleigh velocity can also be expressed in terms of p-wave velocity
    \[ \frac{V_R}{V_p} = \frac{\omega}{\sqrt{\rho K_R}} = \frac{\omega}{\sqrt{\rho A R \sqrt{\lambda + 2\mu / \mu}}} = \alpha K_R \]
    where \[ \alpha = \sqrt{\frac{\mu}{\lambda + 2\mu}} = \sqrt{\frac{1-2v}{2-2v}} \]
  - Values of \( \frac{V_R}{V_p} \) and \( \frac{V_R}{V_s} \) are shown in the figure below

![Figure 5.9 Variation of Rayleigh wave and body wave propagation velocities with Poisson's ratio.](image)

- Rayleigh Wave Displacement Amplitude
  - horizontal and vertical displacements are out-of-phase by 90 degrees
  - this means that the horizontal displacement will be zero when the vertical displacement reaches its maximum
  - the motion produced is a retrograde ellipse
Rayleigh Wave Displacement Amplitude (cont.)

Figure below shows how the amplitude of wave varies with depth (expressed as $z / \lambda$ where $z =$ depth below the surface and $\lambda =$ Rayleigh wave amplitude wavelength)

\[ R = \frac{1}{\sqrt{(V_p/V_R)^2 - 1}} \]

$R =$ minimum epicentral distance
$h =$ focal depth
$V_p =$ p-wave velocity
$V_R =$ Rayleigh wave velocity

Figure 5.10  Horizontal and vertical motion of Rayleigh waves. A negative amplitude ratio indicates that the displacement is in the opposite direction of the surface displacement. (After Richart et al., 1970.)

© Steven F. Bartlett, 2011
3-D Wave Propagation

- Love Waves

  - Love waves are SH waves that are trapped by multiple reflections within the surficial layer.
  
  - Love waves must satisfy the wave equations for s-waves in both surficial layered and the infinite halfspace.

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} &= \begin{cases} 
\frac{G_1}{\rho_1} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) & \text{for } 0 \leq z \leq H \\
\frac{G_2}{\rho_2} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right) & \text{for } z \geq H
\end{cases}
\end{align*}
\]

where \( H \) is the thickness of the surficial layer overlying the infinite halfspace.

- The love wave velocity is given by

\[
\tan \omega H \left( \frac{1}{V_{s_1}^2} - \frac{1}{V_L^2} \right)^{1/2} = \frac{G_2}{G_1} \sqrt{1/v_L^2 - 1/v_{s_2}^2}
\]

where:

- \( H \) - thickness
- \( V_{s_1}, V_L, V_{s_2} \) - shear wave velocities
- \( \rho_1, G_1, \) surficial layer
- \( \rho_2, G_2, V_{s_2} \) - infinite half-space
Love wave (cont.)

- Love wave velocities range from the $s$-wave velocity of the half space (at very low frequencies) to the $s$-wave velocity of the surficial layer (at very high frequencies).

- Because the Love wave velocity, $V_L$, varies as a function of circular frequency, $\omega$, the Love waves are dispersive.

- Dispersive means that the waves in the wave front will disperse (not arrive all at the same time).

- Love wave displacement amplitude
  - Love wave displacement amplitude varies sinusoidally with depth, $H$, in the surficial layer
  - Love wave displacement amplitude decays exponentially with depth in the underlying half space
3-D Wave Propagation

- Love Wave (cont.)

$V(z)$

Variation of particle displacement amplitude with depth for Love waves.

- Dispersion of Surface Waves
  - Rayleigh wave velocity is independent of frequency in a homogeneous half space, thus are not dispersive.
  - Love wave velocity varies with frequency between an upper and lower limit and are dispersive.
  - Rayleigh waves are dispersive in a heterogeneous material. Near the earth's surface, soil and rock stiffness usually increase with depth. Because the depth to which a Rayleigh wave causes significant displacement increases with increasing wavelength, Rayleigh waves in reality are also dispersive.
  - Rayleigh waves of long wavelength (low frequency) can propagate faster than Rayleigh waves of short wavelength (high frequency).
  - Why?
- I-D case - Material Boundary in an Infinite Rod
  - harmonic stress wave traveling in a constrained rod

\[ x_1 \quad M_1 \quad V_1 \quad \rightarrow \quad x_2 \quad M_2 \quad V_2 \]

Incident \hspace{1cm} Reflected \hspace{1cm} transmitted

- incident wave = wave traveling toward interface

\[ \nabla_I (x,t) = \nabla I e^{i(\omega t - k_1 x)} \]
(stress wave)
\[ \lambda_I = 2\pi / k_1 \]
(wave length)

- transmitted wave = wave traveling away from interface

\[ \nabla_T (x,t) = \nabla T e^{i(\omega t - k_2 x)} \]
(stress wave)
\[ \lambda_T = 2\pi / k_2 \]
(wave length)

- reflected wave = wave reflected off of interface

\[ \nabla_R (x,t) = \nabla R e^{i(\omega t + k_1 x)} \]
(stress wave)
\[ \lambda_R = 2\pi / k_1 \]
(wave length)
Waves in a Layered Body (cont.)

- 1-D Case (cont.)
  - Assume that displacement from these stress waves are of the same harmonic form

\[
U_I(x,t) = A_i e^{i(\omega t - k_1 x)}
\]
\[
U_R(x,t) = A_r e^{i(\omega t + k_1 x)}
\]
\[
U_T(x,t) = A_t e^{i(\omega t - k_2 x)}
\]

- Use stress-strain and strain-displacement relations to relate stress amplitude to displacement amplitude

\[
\nabla_I(x,t) = M_i \frac{\partial U_I(x,t)}{\partial x} = -i k_1 M_i A_i e^{i(\omega t - k_1 x)}
\]
\[
\nabla_R(x,t) = M_r \frac{\partial U_R(x,t)}{\partial x} = +i k_1 M_r A_r e^{i(\omega t + k_1 x)}
\]
\[
\nabla_T(x,t) = M_2 \frac{\partial U_T(x,t)}{\partial x} = -i k_2 M_2 A_t e^{i(\omega t - k_2 x)}
\]

- The stress amplitude is related to the displacement amplitude by:

\[
\nabla_I = -i k_1 M_i A_i
\]
\[
\nabla_R = +i k_1 M_r A_r
\]
\[
\nabla_T = -i k_2 M_2 A_t
\]

- At the interface, both compatibility of displacements and continuity of stress must be satisfied

\[
U_I(0,t) + U_R(0,t) = U_T(0,t)
\]
• 1-0 CASE (cont.)

\[
\nabla I (0, t) + \nabla R (0, t) = \nabla T (0, t)
\]

therefore

\[
A_i + A_r = A_t
\]

and

\[
\nabla_i + \nabla_r = \nabla_t
\]

recall that

\[
k M = \omega p v
\]

and

\[
\nabla_i = -i k_1 M_1 A_i
\]

\[
\nabla_r = +i k_1 M_1 A_r
\]

\[
\nabla_t = -i k_2 M_2 A_t
\]

then

\[
-\rho_1 v_1 A_i + \rho_1 v_1 A_r = -\rho_2 v_2 A_t = -\rho_2 v_2 (A_i + A_r)
\]

thus the displacement amplitude of the reflected wave to the incident wave is:

\[
A_r = \frac{\rho_1 v_1 - \rho_2 v_2}{\rho_1 v_1 + \rho_2 v_2} A_i = \frac{1 - \rho_2 v_2/\rho_1 v_1}{1 + \rho_2 v_2/\rho_1 v_1} A_i
\]

the displacement amplitude of the transmitted wave to the incident wave is:

\[
A_t = \frac{2 \rho_1 v_1}{\rho_1 v_1 + \rho_2 v_2} A_i = \frac{2}{1 + \rho_2 v_2/\rho_1 v_1} A_i
\]
Waves in a Layered Body (cont.)

Wednesday, August 17, 2011  12:45 PM

* 1-D CASE (cont.)

Defining the impedance ratio as

\[ \alpha_z = \frac{P_2 v_2}{P_1 v_1} \]

the displacement amplitude can be written as:

\[ A_r = \frac{1 - \alpha_z}{1 + \alpha_z} A_i \]

where \( A_i = -\frac{V_i}{i k M_i} \)

\[ A_t = \frac{2}{1 + \alpha_z} A_i \]

Also, the stresses can be related by:

\[ \sigma_r = \frac{\alpha_z - 1}{1 + \alpha_z} \sigma_i \]

\[ \sigma_t = \frac{2 \alpha_z}{1 + \alpha_z} \sigma_i \]

* \( \alpha < 1 \), wave is approaching softer material
  * reflected wave will have a smaller amplitude than incident wave and its sign will be reversed

* \( \alpha > 1 \), wave is approaching stiffer material
  * transmitted wave will be greater than incident wave amplitude, and the amplitude of the reflected wave will be less than, but have the same sign as the incident wave
1D Case (cont.)

- If $\alpha = 0$ (free-end condition)
  - Zero stress at end condition must be met
  - Displacement of the boundary must be twice displacement of incident wave
  - Reflected wave at the free end has same amplitude as incident wave, but has opposite polarity.

- If $\alpha = \infty$ (fixed-end condition)
  - Zero displacement at end must be met
  - Stress at boundary is twice that of the incident wave.
  - Reflected wave has same amplitude and polarity as incident wave

1D Case - Example Problem

\[ V_s = 400 \text{ m/s} \]
\[ \rho = 1.76 \text{ Mg/m}^3 \]

\[ V_s = 750 \text{ m/s} \]
\[ \rho = 2.24 \text{ Mg/m}^3 \]

Incident wave
- Stress amplitude = 100 kPa
- Frequency = 2 Hz
1-0 CASE - EXAMPLE PROBLEM

\[ \alpha_2 = \frac{(1.76 \text{ kN/m}^2)(400 \text{ m/s})}{(2.24 \text{ kN/m}^2)(750 \text{ m/s})} = 0.419 \]

\[ T_r = \frac{\alpha_2 - 1}{1 + \alpha_2} \text{ (stress amplitude of reflected wave)} \]

\[ T_r = \frac{0.419 - 1}{1 + 0.419} \left(100 \text{ kPa}\right) = -40.5 \text{ kPa} \]

\[ T_t = \frac{2\alpha_2}{1 + \alpha_2} \text{ (stress amplitude of transmitted wave)} \]

\[ T_t = \frac{2(0.419)}{1 + 0.419} \left(100 \text{ kPa}\right) = 53.1 \text{ kPa} \]

\[ \alpha_i = \frac{\alpha_i}{i \omega v_i} \left(\text{displacement amplitude of incident wave}\right) \]

\[ \alpha_i = \frac{i \left(100 \text{ kN/m}^2\right) \left(1.09/3.807 \text{ kN}(5.807 \text{ m/s})\right)}{(2.24 \text{ kN/m}^2)(2\pi)(2.0)(750 \text{ m/s})} \]

\[ \alpha_i = 0.00477 \text{ m} = 4.77 \text{ mm} \]

Note: The i term describes the 90° phase angle between stress and displacement.

\[ \alpha_r = \frac{1 - \alpha_2}{1 + \alpha_2} \alpha_i = \frac{1 - 0.419}{1 + 0.419} \left(4.77 \text{ mm}\right) = 1.95 \text{ mm} \]

\[ \alpha_t = \frac{2}{1 + \alpha_2} \alpha_i = \frac{2}{1 + 0.419} \left(4.77 \text{ mm}\right) = 6.72 \text{ mm} \]
3-D waves (cont.)

Ray path = path that produces minimum travel time.

Ray = vector of ray path at wave front (ray is perpendicular to wave front).

- Ray paths at interfaces
  - Change in ray path at interface of two different materials is known as refraction.
    - Snell's Law
      \[
      \frac{\sin i}{n} = \text{constant}
      \]
      \[
      i = \text{angle between the ray path and the normal to the interface}
      \]
      \[
      n = \text{velocity of } p \text{ or } s \text{ wave}
      \]
3-D waves (cont.)

Snell's Law (cont.)

\[
\frac{\sin i_1}{V_1} = \frac{\sin i_2}{V_2}
\]

incident wave, \( V_1 \)

refracted ray path, \( V_2 \)

material 1

material 2

\[
\frac{\sin i_1}{V_1} = \frac{\sin i_1}{V_1}
\]

incident wave, \( V_1 \)

reflected ray path, \( V_1 \)
* Ray paths at interfaces (cont.)*

![Diagram showing ray paths at interfaces](image)
- Ray paths and Snell's Law (cont.)

<table>
<thead>
<tr>
<th>Wave Type</th>
<th>Velocity</th>
<th>Angle w/ normal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Incident p</td>
<td>U</td>
<td>a</td>
</tr>
<tr>
<td>Incident s</td>
<td>V</td>
<td>b</td>
</tr>
<tr>
<td>Reflected p</td>
<td>U</td>
<td>c</td>
</tr>
<tr>
<td>Reflected s</td>
<td>V</td>
<td>d</td>
</tr>
<tr>
<td>Refracted p</td>
<td>Y</td>
<td>e</td>
</tr>
<tr>
<td>Refracted s</td>
<td>Z</td>
<td>f</td>
</tr>
</tbody>
</table>

Using Snell's Law

\[
\frac{\sin c}{U} = \frac{\sin b}{V} = \frac{\sin c}{U} = \frac{\sin d}{V} = \frac{\sin e}{Y} = \frac{\sin f}{Z}
\]

Note: Snell's Law does not say anything about the amplitude of the refracted and reflected waves. The amplitude of the above waves are all different.

- Near vertical refraction of SH-wave

```
  \[
  V_p = 500 \text{ ft/s} \\
  V_p = 1,000 \text{ ft/s} \\
  V_p = 1,500 \text{ ft/s} \\
  V_s = 2,000 \text{ ft/s} \\
  V_s = 2,500 \text{ ft/s}
  \]
```

Refractive of SH-wave through layers of progressively softer material. This behavior is important for 1-D response analysis.
* Near vertical refraction of SH-wave (cont.)
  
  The 1-D ground response program called SHAKE assumes vertically propagating SH-waves.

* Reasonable approximation for sites that are moderate to great distance from causative fault.

Note: Nearly vertical waves at site.

* May not be a reasonable assumptions for sites close to the fault or for faults that dip under the site.

Note: Inclined waves arrive at site.

Note: Fault rupture is directly under site.
• Attenuation of stress waves
  • Idealized homogeneous, isotropic, elastic material has no decrease in wave amplitude with distance from source
  • Real materials, especially soils, are not elastic materials.
  • Damping causes amplitude of stress wave to decrease with distance from the source. This is called attenuation.

• Two types of damping:
  • material
  • geometric (radiation)

• Material Damping
  • elastic energy of wave is converted to heat
  • conversion of some energy to heat decreases amplitude of wave
  • viscous damping is often used to represent the dissipation of elastic energy
  • visco-elastic wave propagation
    • Kelvin–Voigt solid

\[ T = G Y + \eta \frac{dY}{dt} \]
• Visco-elastic model (cont.)

\[ \tau_{xz} = \tau = \text{shear stress} \]
\[ \gamma = \frac{du}{dz} = \text{shear strain} \]

• Elastic spring part \((\tau \propto \gamma)\)
• Viscous part \((\tau \propto \dot{\gamma})\) \((\tau \propto \text{strain rate})\)
• For harmonic shear strain

\[ \gamma = \gamma_0 \sin \omega t \]

then

\[ \tau = G\gamma_0 \sin \omega t + \omega \eta \gamma_0 \cos \omega t \]

\(\tau\) elastic part \(\gamma\) viscous part

• Stress-strain loop for viscoelastic material
  • Loop is an ellipse
• Energy dissipate during one loop or cycle is a function of damping
  • Energy loss \(\Delta W\)

\[ \Delta W = \int_{t_0}^{t_0 + 2\pi/\omega} \tau \frac{d\gamma}{dt} \, dt \]

\[ \Delta W = \frac{\pi - \eta \omega}{2} \gamma_0^2 \]
Stress strain loop for viscoelastic material (cont.)

- Energy dissipated in a single cycle is given by the area of the ellipse.

\[ \text{Area within loop} = \Delta W \]

\[ \xi = \text{damping} = \frac{1}{4\pi} \frac{\Delta W}{W} = \frac{1}{4\pi} \frac{\pi \omega \gamma_0^2}{\frac{1}{2} G \gamma_0^2} = \frac{\omega}{2G} \]

where \( W \) = peak energy during one cycle

\[ W = \frac{1}{2} G \gamma_0^2 \]

\[ \xi = \frac{\omega}{2G} \quad \text{damping} \]

Note that the above equation suggests that damping is proportional to the circular frequency. This is not very desirable from a computational standpoint.

Solids dissipate their energy by grain slippage and their damping is somewhat insensitive to frequency.
• Damping (cont.)

  To eliminate the frequency dependency while still using the viscoelastic formulation the equation

  \[ \frac{\varepsilon}{\varepsilon_0} = \frac{\nu}{2G} \]

  is often rearranged to produce an equivalent viscosity that is inversely proportional to frequency

  \[ \eta = \frac{2G}{\nu} \varepsilon \]

  Thus

  \[ \varepsilon = \frac{\frac{2G}{\nu} \varepsilon \cdot \eta}{2G} \]

  this means \( \varepsilon \) is now independent of \( \nu \) and of \( G \) for the viscous part of the model.

• 1-D wave equation

  \[ \rho \frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial z^2} \]

  Substituting \( \tau = \sqrt{\alpha z} \) \( \xi = du/dz \) adding in the viscous part and differentiating:

  \[ \rho \frac{\partial^2 u}{\partial t^2} = G \frac{\partial^2 u}{\partial z^2} + \eta \frac{\partial^3 u}{\partial z^2 \partial t} \]
Finite difference calculation loop written with differential calculus

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \]

\[ \Delta t \]

\[ \Delta x \]

\[ \frac{\partial v_x}{\partial t} = \frac{\partial^2 v_x}{\partial x^2} \]

\[ A_x = \frac{\partial v_x}{\partial t} \]

\[ \frac{\partial F_x}{m} = A_x \]

\[ \Delta F_x = \frac{\partial v_x}{\partial y \partial z} \]

\[ V_x = \text{velocity } x\text{-direction} \]

\[ \varepsilon_x = \text{strain } x\text{-direction} \]

\[ E = \text{modulus } x\text{-direction} \]

\[ \sigma_x = \text{normal stress } x\text{-direction} \]

\[ F_x = \text{force in } x\text{-direction} \]

\[ a_x = \text{acceleration } x\text{-direction} \]

\[ m = \text{mass of unit volume} \]

Finite difference calculation loop written with incremental approach

\[ \frac{\partial v_x}{\partial t} = v_x \]

\[ \frac{\partial u}{\partial x} = \varepsilon_x \]

\[ \frac{\partial v}{\partial \varepsilon_x} = \frac{\partial v}{\partial x} \]

\[ a_x = \frac{\Delta v}{\Delta t} \]

\[ F_x = \frac{\Delta v}{\Delta y \Delta z} \]

\[ \alpha_x = \text{spacing between nodes} \]

\[ \alpha_y, \alpha_z = \text{unit lengths in } y, z \text{ coord directions} \]
Note that with this approach we can approximate the change of things that vary either in space or time, or both. In regards to time, we will use the forward differencing approach.

© Steven F. Bartlett, 2011
• Divide domain into \( N+1 \) nodes \( i \) sublayers

\[
\begin{array}{c|c}
\text{Nodes} & \text{Sublayers} \\
\hline
N+1 & 1 \\
N & \cdots \\
i+1 & i+1 \\
i & i \\
2 & 2 \\
1 & 1 \\
\end{array}
\]

• Input cosine wave at node 1 (base) as a velocity input. This approach represents a rigid base moving back and forth as a function of the cosine wave

\[
v_{ix} = A \cos(\omega t + \phi) \tag{3}
\]

• Calculate the change in position for time increment \( \Delta t \)

\[
u_{ix}, t + \Delta t = u_{ix}, t + \Delta t \tag{4}
\]

• Calculate the shear strain induced by this change in position during \( \Delta t \)

\[
\gamma_{ix}, t + \Delta t = \frac{(u_{ix}, t + \Delta t - u_{ix+1}, t + \Delta t)}{\Delta z} \tag{6}
\]
• Calculate the shear stress in the layer using the shear strain and shear modulus, $G$

$$\tau_{i, t+\Delta t} = G \cdot \gamma_{i, t+\Delta t}$$  \hspace{1cm} (7)

• This unbalanced shear stress causes the top node of the layer to move with a new velocity. The velocity can be calculated from the wave equation.

$$\frac{\partial \tau}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2} = \rho \frac{\partial \dot{u}}{\partial t}$$  \hspace{1cm} (8)

Written in an incremental approach, this is:

$$\frac{\Delta \tau}{\Delta z} = \frac{\tau_{i, t+\Delta t} - \tau_{i+1, t+\Delta t}}{\Delta z}$$  \hspace{1cm} (9)

$$\frac{\Delta \dot{u}}{\Delta t} = \frac{\Delta V}{\Delta t} = \frac{V_{i, t+\Delta t} - V_{i, t}}{\Delta t}$$  \hspace{1cm} (10)

Substituting (9) and (10) into (8) produces for the top node

$$V_{i+1, t+\Delta t} = V_{i+1, t} + \Delta t \left( \frac{\tau_{i, t+\Delta t} - \tau_{i+1, t+\Delta t}}{\rho \Delta z} \right)$$  \hspace{1cm} (11)

This is used to calculate the new velocity in the top node of layer $i$. 

© Steven F. Bartlett, 2011
* Once we have solved for the new velocity of node \( i+1 \), we are ready to start another time step, thus

\[ \begin{align*}
V_{i, t+1} & = V_{i, t} + \frac{\Delta t}{\Delta x} (u_{i+1, t} - u_{i, t}) \\
\end{align*} \]

for the next time step.

* Boundary condition considerations

  * free end (top of domain)
    * \( y = 0 \) \[ \Rightarrow \] layer \( N \)
    * \( z = 0 \)
    * \( V_{z, t+1} \text{ (top node)} = 2 \times \frac{\text{Eq}(11)}{N+1} \)
    * 2x factor req'd for doubling of wave at free surface.

  * base end (bottom of domain)
    * \( V_{z} \) of any time \( t \) is determined from cosine forcing function.
Solution of Wave Equation for Shear Wave (cont.)

Wednesday, March 05, 2014

Time step: 0.01

\( v(t) = A \cos(\omega t + \phi) \)

Amplitude: 0.300

Angular frequency: 6.283

Phase angle: 0.000

Phase velocity: 0.05

Poisson's Ratio: 0.35

Elapsed time: 5.010

Total run time: 5.000

Nodes:

\( \begin{array}{ccccccc}
\text{Depth} & v_{i,t} & u_{i,t} & v_{i+1,t} & u_{i+1,t} & \Delta u_{i,t} & \Delta v_{i,t} \\
\text{Layer} & (1) & (2) & (3) & (4) & (5) & (6) \\
1 & 0 & 3.17E-01 & -9.57E-01 & 6.87E-01 & -9.54E-01 & 3.17E-03 & 0.00E+00 & 0.00E+00 \\
2 & -1 & 3.51E-01 & -9.46E-01 & 7.45E-01 & -9.42E-01 & 3.51E-03 & 1.16E-02 & 3.70E+02 \\
3 & -2 & 3.40E-01 & -9.10E-01 & 7.06E-01 & -9.06E-01 & 3.40E-03 & 3.62E-02 & 1.16E+02 \\
4 & -3 & 2.47E-01 & -8.49E-01 & 5.61E-01 & -8.47E-01 & 2.47E-03 & 5.92E-02 & 1.90E+02 \\
5 & -4 & 9.60E-02 & -7.68E-01 & 3.97E-01 & -7.66E-01 & 9.60E-04 & 7.66E-02 & 2.52E+02 \\
7 & -6 & 1.51E-01 & -5.57E-01 & 3.84E-01 & -5.55E-01 & 1.51E-03 & 1.15E+02 & 3.70E+02 \\
8 & -7 & 2.25E-01 & -4.27E-01 & 3.70E-01 & -4.25E-01 & 2.25E-03 & 1.30E+01 & 4.16E+02 \\
9 & -8 & 2.53E-01 & -2.88E-01 & 3.46E-01 & -2.86E-01 & 2.53E-03 & 1.39E-01 & 4.45E+02 \\
10 & -9 & 3.32E-01 & -1.44E-01 & 3.65E-01 & -1.41E-01 & 3.32E-03 & 1.45E-01 & 4.64E+02 \\
11 & -10 & 3.00E-03 & 5.99E-03 & 2.99E-03 & 1.47E+01 & 4.70E+02 \\
\end{array} \)

Rigid base:

-10 2.99E-01 From Eq. (1)

Macros:

- Ctrl + a: Resets all back to t = 0
- Ctrl + q: Runs analysis

\( \Delta t_{\text{max}} = 0.025 \) (maximum time step should not exceed this)

Notes:

Step (1) & (2): Velocity and displacement at beginning of time step
Step (3): \( \Delta u_{i,t} = v_{i,t} - \Delta t \) (change in displacement during time step)
Step (4): \( u_{i+1,t} = u_{i,t} + \Delta u_{i,t} \) (new position at end of time step)
Step (5): \( v_{i+1,t} = \left( v_{i,t} + \left( \frac{\Delta t}{\Delta z} \right) \right) \) (shear strain during time step - forward differencing)
Step (6): \( \tau_{i+1,t} = G \gamma_{i+1,t} \) (shear stress during time step)
Step (7): \( v_{i+1,t+1} = v_{i+1,t} + \Delta t / (\Delta z / s) \) (velocity in stress wave - forward differencing)
Step (8): \( v_{i,t+1} \) and \( u_{i,t+1} \) become \( v_{i,t} \) and \( u_{i,t} \) for next time step (copy/paste done by macro)
\( \text{FLAC verification of solution} \)

\[
\text{config dynamic extra 5} \\
\text{grid 1 10} \\
\text{model elastic} \\
\text{ini y mul 1} \\
\text{; set dy\_damp rayl 0.05 5; 5 percent damping at 5 hz} \\
\text{fix y} \\
\text{prop dens 2000 bulk 9.6E6 shear 3.2E6} \\
\text{def wave} \\
\text{ wave=amp*cos(omega*dytime)} \\
\text{ if dytime>=100} \\
\text{ wave=0} \\
\text{ endif} \\
\text{ end} \\
\text{ set amp=0.3} \\
\text{ set omega = 6.283} \\
\text{ apply xvel 1 hist wave yvel=0 j=1} \\
\text{ his 1 xdisp i 1 j 1} \\
\text{ his 2 xdisp i 1 j 11} \\
\text{ his 3 xvel i 1 j 1} \\
\text{ his 4 dytime} \\
\text{ set dytime = 0} \\
\text{ history 999 unbalanced} \\
\text{ solve dytime 5.01} \\
\text{ save model2.sav 'last project state'}